

# Dilated Matrix Inequalities for Control Design in Systems with Actuator Constraint\*

Solmaz Sajjadi-Kia and Faryar Jabbari<sup>†</sup>

## Abstract

In this paper, we present a new variation of dilated matrix inequalities (MIs) for Bounded Real MI, invariant set MI and constraint MI, for both state and output feedback synthesis problems. In these dilated MIs, system matrices are separated from Lyapunov matrices to allow the use of different Lyapunov matrices in multi-objective and robust problems. To demonstrate the benefit of these new dilated MIs over conventional ones, they are used in solving controller synthesis problem for systems with bounded actuator in disturbance attenuation. It is shown that for the resulting multi-objective saturation problem, the new form of dilated MIs achieves an upper bound for  $L_2$  gain that is less than or equal to the upper bound estimate achieved by conventional method.

## 1 INTRODUCTION

Most of the (Linear) Matrix Inequality ((L)MI) characterizations in control techniques such as  $H_\infty$ ,  $H_2$ , use a quadratic Lyapunov function ( $V = x^T P x$   $|P > 0$ ) to develop their MIs (e.g., see ref [1]). The resulting MIs end up having entries with the products of the Lyapunov variables and system matrices. This causes some degree of conservatism in multi-objective and robust problems by forcing common Lyapunov matrices for all objectives. For example, see Ref. [2]-[4] which make use of common Lyapunov variable in their multi-objective problems.

Recently, researchers have been using matrix dilation results to reduce this conservatism. The earlier, and the best known, of the results obtained by these new techniques are in discrete time settings ([6],[7]). Although a lot of effort has also been made for the continuous-time case, it is still an open problem, mostly due to the fact that dilations to reduce conservatism destroy the convexity in some important cases. Some of the very nice and convex results in continuous time are achieved by Ebihara et al. in [8]-[10], in case where the problem can be case as

$$AX + XA^T + \delta_1 X + \delta_2 AXA^T + X\Delta^T\Delta X < 0 \quad (1)$$

for a suitable choice of  $X$ ,  $\Delta$ , etc. By assigning different matrices to  $A$ ,  $\delta_1$ ,  $\delta_2$  and  $\Delta$ , this general form covers some continuous-time control problems such as stability,  $H_2$  and D-stability. Furthermore, references [15] and [16] present a technique that, using a projection (or Elimination lemma) based approach, leads to set of convex search for several important problems;  $H_2$ , stability, D-stability, etc.

Unfortunately, neither of the two approaches above deal with synthesis inequalities faced in the  $L_2$ -gain (i.e., bounded real problems) or several invariant set determination problems. Roughly, these problems result in a constant term in (1). There have been dilated MIs for such cases, as well ([11]-[13], among others). So far, there seems to be two weaknesses associated with these set of results. Often, the synthesis results are for the full state case, by exploiting a

\*This work was supported by NSF Grant CMS-0510874

<sup>†</sup>The authors are with the Department of Mechanical and Aerospace Engineering, University of California, Irvine, Irvine, CA 92697, USA. [ssajjadi](mailto:ssajjadi@uci.edu), [fjabbari](mailto:fjabbari@uci.edu)

structure that holds only in full state problem. Furthermore, they all seem to need an additional scalar variable which enters the MI in a non-convex fashion ([11]-[13]). While it seems that such a non-convexity is inevitable, it can be addressed with a line search since the culprit is a scalar variable.

Here, we present a new dilated MI, that can be applied to the bounded real LMI, as well as to the matrix inequalities that are used in the invariant set for peak bounded disturbance ([1]) and the constraint LMI ([1]). These new MIs are obtained explicitly through a constructive methods, to avoid the ambiguities that can – at time – accompany results based on the Projection Lemma (see [8] for a discussion on this issue). We also rely on a scalar variable that renders the problem non-convex and use line searches to obtain the final result. Fortunately, we show that the proposed approach is rather easily extended to the output feedback problem, assuming the controller is of the same order as the plant.

Finally, we study the effect of this new matrix dilation technique in reducing conservatism in controller design for linear systems with bounded actuators which avoid saturation. As mentioned in [2], this is often set as a multi-objective problem, and can suffer the conservatism forced by common Lyapunov matrix. We show that, these new MIs lead us to a problem whose performance is at least equal to the one with standard MIs.

The system we study has the standard model

$$\begin{cases} \dot{x}_p = Ax_p + B_1w + B_2u \\ z = C_1x_p + D_{11}w + D_{12}u \\ y = C_2x_p + D_{21}w \end{cases} \quad (2)$$

with the closed loop

$$\begin{cases} \dot{x} = A_{cl}x + B_{cl}w \\ z = C_{cl}x + D_{cl}w \end{cases} \quad (3)$$

where the details differ in the state feedback and output feedback cases. The Transfer function of this system is  $T_{zw}(s) = C_{cl}(sI - A_{cl})^{-1}B_{cl} + D_{cl}$ . We use  $\mathbf{He}(A)$  as short notation for  $A + A^T$  to save space. The rest of the notations throughout the paper follow standard practices.

## 2 Dilated MIs for Some Practical Design Specifications

In this section, we derive equivalent dilated MIs for standard Bounded real LMI, invariant set MI and constraint LMI. In the new MIs, the system matrices and the Lyapunov variable are decoupled.

### 2.1 A Dilated LMI for $L_2$ Gain

**Lemma 1 (Bounded Real Lemma [1])**  $A_{cl}$  is stable with  $\|T_{zw}(s)\|_\infty < \gamma_{con}$  if and only if  $\bar{\sigma}(D_{cl}) < \gamma_{con}$  and there exists  $Q_1 > 0$  such that

$$\begin{pmatrix} A_{cl}Q_1 + Q_1A_{cl}^T & B_{cl} & Q_1C_{cl}^T \\ \star & -\gamma_{con}I & D_{cl}^T \\ \star & \star & -\gamma_{con}I \end{pmatrix} < 0 \quad (4)$$

**Theorem 1** The closed-loop system (3) is stable and its  $L_2$  gain is less than  $\gamma_{new}$  if there exist a positive constant  $0 < \epsilon_1 < 1$ , and square matrices  $X_1 > 0$  and  $G_1$  which satisfy:

$$\begin{pmatrix} X_1 & B_{cl} & 0 & -X_1 \\ \star & -\gamma_{new}I & D_{cl}^T & 0 \\ \star & \star & -\gamma_{new}I & 0 \\ \star & \star & \star & 0 \end{pmatrix} + \mathbf{He}(\mathcal{Q}^T G_1 \mathcal{P}) < 0 \quad (5)$$

where

$$\mathcal{P} = [I \quad 0 \quad 0 \quad -2\epsilon_1 I]; \quad \mathcal{Q} = [(A_{cl}^T - \frac{1}{2}I) \quad 0 \quad C_{cl}^T \quad I].$$

**Proof 1** Proof is through showing MIs (4) and (5) are equivalent. Suppose that MI (5) holds. Consider the explicit bases of nullspaces of  $\mathcal{P}$  and  $\mathcal{Q}$

$$\mathcal{N}_{\mathcal{P}} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \frac{1}{2\epsilon_1}I & 0 & 0 \end{pmatrix}, \quad \mathcal{N}_{\mathcal{Q}} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ -A_{cl}^T + \frac{1}{2}I & 0 & -C_{cl}^T \end{pmatrix}$$

By multiplying  $\mathcal{N}_{\mathcal{Q}}$  and its transpose from the right and left sides respectively, and considering  $\mathcal{Q}\mathcal{N}_{\mathcal{Q}} = 0$ , inequality (5) becomes

$$\begin{pmatrix} A_{cl}X_1 + X_1A_{cl}^T & B_{cl} & X_1C_{cl}^T \\ \star & -\gamma_{new}I & D_{cl}^T \\ \star & \star & -\gamma_{new}I \end{pmatrix} < 0 \quad (6)$$

Now if we define  $X_1 = Q_1$  and  $\gamma_{new} = \gamma_{con}$ , it is clear that (4) holds. Next, multiplying (5) from right and left by  $\mathcal{N}_{\mathcal{P}}$  and its transpose respectively gives

$$\begin{pmatrix} (1 - \frac{1}{\epsilon_1})X_1 & B_{cl} & 0 \\ \star & -\gamma_{new}I & D_{cl}^T \\ \star & \star & -\gamma_{new}I \end{pmatrix} < 0 \quad (7)$$

This inequality implies that (5) can have a solution only for  $0 < \epsilon_1 < 1$ .

On the other hand, suppose that (4) holds with  $Q_1 > 0$ . Note that (4) can be rewritten as (6) by defining  $Q_1 = X_1$  and  $\gamma_{con} = \gamma_{new}$ . Since  $X_1 > 0$ , for any  $\bar{\epsilon}_1 > 0$  we have

$$\mathcal{R}^T (4\bar{\epsilon}_1 X_1)^{-1} \mathcal{R} \geq 0 \quad (8)$$

where  $\mathcal{R} = [-2\bar{\epsilon}_1 X_1 (A_{cl}^T - \frac{1}{2}I) \quad 0 \quad -2\bar{\epsilon}_1 X_1 C_{cl}^T]$ . Since the right hand side of the inequality is of order  $\bar{\epsilon}_1$ , it is possible to find a sufficiently small  $\bar{\epsilon}_1 > 0$  which for this  $\bar{\epsilon}_1$ , and any  $\epsilon_1 < \bar{\epsilon}_1$ , the following holds

$$[\text{left side of (6)}] + [\text{left side of (8)}] < 0.$$

Applying the Schur complement to this inequality leads to

$$\begin{pmatrix} \Pi & B_{cl} & X_1 C_{cl}^T & -2\epsilon_1 (A_{cl} - \frac{1}{2}I) X_1 \\ \star & -\gamma_{new}I & D_{cl} & 0 \\ \star & \star & -\gamma_{new}I & -2\epsilon_1 C_{cl} X_1 \\ \star & \star & \star & -4\epsilon_1 X_1 \end{pmatrix} < 0$$

where  $\Pi = \mathbf{He}((A_{cl} - \frac{1}{2}I)X_1) + X_1$ . By choosing  $G_1 = G_1^T = X_1$ , this inequality can be written as (5); i.e., satisfaction of (4) leads to a specific choice for the matrices  $G_1$  and  $X_1$  that satisfy (5), with the same performance estimate.

## 2.2 A Dilated MI for the Invariant set

**Lemma 2 ( Invariant Set MI [1])**  $\mathcal{E} = \{x | x^T P x < w_{max}^2\}$  is a reachable set (invariant set) for the LTI system (3) exposed to a peak bounded disturbance  $w(t)^T w(t) \leq w_{max}^2$  if there exist a scalar  $\alpha_{con} > 0$  and  $Q_2 = P^{-1} > 0$  such that the following MI is feasible

$$\begin{pmatrix} A_{cl}Q_2 + Q_2A_{cl}^T + \alpha_{con}Q & B_{cl} \\ \star & -\alpha_{con}I \end{pmatrix} < 0 \quad (9)$$

**Theorem 2** Inequality (9) is feasible if and only if there exist a square matrix  $G_2$ , a constant  $0 < \epsilon_2 < 1$  and  $X_2 = P^{-1} > 0$  which satisfy the following condition for some  $\alpha_{new} > 0$ :

$$\begin{pmatrix} X_2 & B_{cl} & -X_2 \\ \star & -\alpha_{new}I & 0 \\ \star & \star & 0 \end{pmatrix} + \mathbf{He}(\mathcal{Q}^T G_2 \mathcal{P}) < 0 \quad (10)$$

where  $\mathcal{P} = [I \ 0 \ -2\epsilon_2 I]$  and  $\mathcal{Q} = [(\hat{A}^T - \frac{1}{2}I) \ 0 \ I]$ , with  $\hat{A} = A_{cl} + \frac{\alpha_{new}}{2}I$ .

**Proof 2** Proof is similar to the Theorem 1. In this case  $\mathcal{R}$  in inequality (8) is

$$\mathcal{R} = [-2\epsilon_2 X_2 (\hat{A}^T - \frac{1}{2}I) \ 0]$$

### 2.3 A Dilated MI for Constraint LMI

Constraint LMI ([1]): Suppose that  $y = Kx$  and  $x \in \{x | x^T P x < w_{max}^2\}$ . Then,  $\|y\|^2$  will be less than or equal to  $u_{lim}^2$  if the following LMI holds for  $Q_3 = P^{-1}$

$$\begin{pmatrix} -Q_3 & -Q_3 K^T \\ \star & -\frac{u_{lim}^2}{\omega_{max}^2} I \end{pmatrix} < 0 \quad (11)$$

**Theorem 3** The matrix inequality (11) is feasible if and only if the following inequality is feasible for some  $0 < \epsilon_3 < 1$  and  $X_3 > 0$ :

$$\begin{pmatrix} X_3 & 0 & -X_3 \\ \star & -\frac{u_{lim}^2}{\omega_{max}^2} I & 0 \\ \star & \star & 0 \end{pmatrix} + \mathbf{He}(\mathcal{Q}^T G_3 \mathcal{P}) < 0 \quad (12)$$

where  $\mathcal{P} = [I \ 0 \ -2\epsilon_3 I]$  and  $\mathcal{Q} = [-I \ -K^T \ I]$ .

**Proof 3** Proof is similar to the Theorem 1. In this case  $\mathcal{R}$  in inequality (8) is

$$\mathcal{R} = [2\epsilon_3 X_3 \ 2\epsilon_3 X_3 K^T]$$

## 3 A multi-objective problem solved using the new dilated MIs

In this section, the new MIs are used to solve a multi-objective problem. The problem considered here is that of controller design for a system with bounded actuators exposed to a peak bound disturbance,  $w(t)^T w(t) < w_{max}^2$ . Nevertheless, the idea can be applied to other multi-objective cases. The goal is to design a controller which makes this system internally stable and guarantees disturbance attenuation, while avoiding any violation of saturation limit ( $u_{lim}$ ). As mentioned in [2], this problem is a multi-objective problem, often yielding very conservative results. Typically, the linear or low gain controller for this problem is based on finding a controller (a state feedback or dynamic output feedback compensator) which gives the best  $L_2$  performance while making sure that saturation limits are not violated, by keeping the maximum of the control input below the limit for all the points in the reachable set of the closed loop system. We can formulate the proposed solution through the following two algorithms:

- **Algorithm 1** (Conventional approach): For common  $Q_1 = Q_2 = Q_3 = Q > 0$ , minimize  $\gamma_{con}$  in LMI (4) subject to MIs (9) and (11).

- **Algorithm 2** (New approach using new dilated MIs): For  $X_1 > 0$ ,  $X_2 = X_3 > 0$  and common  $G_1 = G_2 = G_3$ , minimize  $\gamma_{new}$  in (5) subject to MIs (10) and (12).

In Algorithm 2, as we show later in both state feedback and full order output feedback cases, common  $G$  is needed to turn the MIs into appropriate form to get a unique controller. However, there is no obligation to use a common Lyapunov matrices for the  $L_2$  and invariant set inequalities. The reason to take  $X_2 = X_3$  in Algorithm 2 is that we are keeping the maximum of the controller, below its limit, for all the points in the reachable set of the closed-loop system. Here, the reachable set is  $x^T X_2^{-1} x < w_{max}^2$ .

The following theorem states the advantage of the new approach expressed through Algorithm 2 over the conventional one obtained by Algorithm 1, by guaranteeing better (or at least no worse)  $L_2$  gain estimate.

**Theorem 4 (Multi-objective)** *For the multi-objective saturation problem mentioned above, Algorithm 2 with a common auxiliary variable,  $G$ , but with non-common Lyapunov variables, always achieves an upper bound estimate for the  $L_2$  gain that is less than or equal to  $L_2$  gain performance estimate achieved by Algorithm 1.*

**Proof 4** *If Algorithm 1 is solved, then Theorem 1 implies that there is a positive  $\epsilon$  such that for any  $\epsilon_1 < \epsilon$  by taking  $X_1 = G = G^T = Q$  and  $\gamma_{new} = \gamma_{con}$ , we can satisfy (5) with the same closed-loop system derived by solving Algorithm 1. Similarly, following the same argument, Theorems 2 and 3 guarantee that there are  $\epsilon_2$  and  $\epsilon_3$  such that  $X_2 = G = G^T = Q$  and  $\alpha_{new} = \alpha_{con}$  satisfy (10) and (12). Therefore, all the MIs in Algorithm 2 are feasible with  $\gamma_{new} = \gamma_{con}$  if we set  $G = G^T = X_1 = X_2 = Q$  and use the same closed-loop system, as obtained by Algorithm 1.*

*Therefore, any solution of Algorithm 1 can be achieved by Algorithm 2, for a small enough  $\epsilon$ 's, without exploiting the ability to use different Lyapunov matrices, which could only improve the results.*

Recall that if MI (5) holds for some  $\epsilon_1$ , it would hold for any  $\epsilon < \epsilon_1$ . Same argument is true for MIs (10) and (12). Therefore, using the same  $\epsilon$  still allows the results to be at least as good as those from Algorithm 1. Therefore in Algorithm 2, to decrease the computational cost of line search for  $\epsilon_1$ ,  $\epsilon_2$ , and  $\epsilon_3$ , we can use the same  $\epsilon$  for all MIs. This leads to some degree of conservatism. The conservatism is due to the fact that the best result obtained by Algorithm 2 is not necessarily preserved if we lower all or any of the  $\epsilon_i$ . For best results, the  $\epsilon$ 's are allowed to vary independently (see the numerical examples).

In Section 3.1, Algorithms 1 and 2 are applied to state-feedback case and in Section 3.3, these algorithms are used for full order dynamic output feedback compensator. The result of these two algorithms are compared through numerical examples.

### 3.1 State Feedback Case

In this section, state feedback controller synthesis is considered. Therefore,  $u = Kx_p$ , and the closed-loop system in (3) can easily be obtained with  $x = x_p$ . For state feedback control, Algorithm 1 and 2 can be expressed in the convex MI set as in lemmas below.

**Lemma 3 ([2])** *System (3) with  $w(t)^T w(t) \leq w_{max}^2$  and state feedback  $K$  is internally stable, never saturates and has a disturbance attenuation level  $\gamma_{con}$  if there exist  $Q > 0$ ,  $Y$  and a positive constant  $\alpha_{con} > 0$  such that*

$$\begin{pmatrix} \Pi & B_1 & QC_1 + Y^T D_{12}^T \\ \star & -\gamma_{con} I & D_{11}^T \\ \star & \star & -\gamma_{con} I \end{pmatrix} < 0$$

$$\begin{pmatrix} \Pi + \alpha_{con} Q & B_1 \\ \star & -\alpha_{con} I \end{pmatrix} < 0$$

$$\begin{pmatrix} -Q & -Y^T \\ \star & -\frac{u_{lim}^2}{\omega_{max}^2} I \end{pmatrix} < 0$$

where  $\Pi = AQ + QA^T + B_2 Y + Y^T B_2$ . The variables in this problem are  $Q$ ,  $\gamma_{con}$ ,  $\alpha_{con}$  where  $\alpha_{con}$  is searched through a line search. The controller is given by  $K = YQ^{-1}$ .

**Lemma 4** System (3) with  $w(t)^T w(t) \leq w_{max}^2$  and state feedback  $K$  is internally stable, never saturates and has a disturbance attenuation level  $\gamma_{new}$  if there exist  $X_1 > 0$ ,  $X_2 > 0$ ,  $Y$ , square matrix  $G$ , constant  $\alpha_{new} > 0$ , and small positive scalars  $\epsilon_i < 1$  ( $i = 1, 2, 3$ ) such that

$$\begin{pmatrix} X_1 + \Pi + \Pi^T & B_1 \\ \star & -\gamma_{new} I \\ \star & \star \\ \star & \star \\ G^T C^T + Y^T D_{12}^T & -X_1 + G^T - 2\epsilon_1(\Pi) \\ D_{11}^T & 0 \\ -\gamma_{new} I & -2\epsilon_1(CG + D_{12}Y) \\ \star & -2\epsilon_1(G^T + G) \end{pmatrix} < 0$$

$$\begin{pmatrix} X_2 + \Pi + \Pi^T + \frac{\alpha_{new}}{2}(G + G^T) & B_1 \\ \star & -\alpha_{new} I \\ \star & \star \\ -X_2 + G^T - 2\epsilon_2(\Pi + \frac{\alpha_{new}}{2}G) & \\ 0 & \\ -2\epsilon_2(G^T + G) & \end{pmatrix} < 0$$

$$\begin{pmatrix} X_2 - G - G^T & -Y^T & -X_2 + G^T - 2\epsilon_3 G \\ \star & -\frac{u_{lim}^2}{\omega_{max}^2} I & 2\epsilon_3 Y \\ \star & \star & -2\epsilon_3(G^T + G) \end{pmatrix} < 0$$

where  $\Pi = AG + B_2 Y - \frac{1}{2}G$ . The variables in this problem are: Lyapunov matrices  $X_1$  and  $X_2$  as well as  $\gamma_{new}$ ,  $Y$ ,  $\alpha_{new}$  and  $\epsilon_i$  ( $i = 1, 2, 3$ ) where  $\alpha_{new}$  and  $\epsilon$ 's are searched through line search. The controller is given by  $K = YG^{-1}$ .

In this lemma, same slack variable  $G$  ( $G_1 = G_2 = G_3 = G$ ) is used to make the MI set convex and to get a unique solution for  $K$ .

### 3.2 Numerical Example: State Feedback

In this section we use the results of Section 3.1 in a numerical example. Consider the example from [13]

$$\left( \begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \end{array} \right) = \left( \begin{array}{cccc|c|c} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -k & k & -f & f & 0 & 1 \\ k & -k & f & -f & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.01 \end{array} \right) \quad (13)$$

with  $k = 2$  and  $f = 0.2$ . The system is exposed to a peak bounded disturbance with  $w_{max} = 5$ . The controller limit is  $u_{lim} = 8$ . Using Algorithm 1, we get the following controller

$$K = [ -1.7970 \quad -0.7094 \quad -2.2916 \quad -2.1091 ]$$

which gives the minimum  $\gamma_{con}^* = 1.3038$ . This problem is also solved through Algorithm 2. To decrease the computational cost, we used a same  $\epsilon$  in all three dilated MIs. Based on the simulations that we have done so far, the variation of  $\gamma_{new}$  with  $\epsilon$  is bowl shaped. Therefore, instead of a full line search on  $\epsilon$ , we used the Golden-ratio method to identify the minimum performance,  $\gamma_{new}^*$ , and the corresponding  $\epsilon$ . The result is achieved very fast (9 iteration). The best performance that can be achieved by new method is  $\gamma_{new}^* = 0.8345$  at  $\epsilon = 0.0802$ , which is about 36% improvement, over the conventional method. The controller associate with  $\gamma_{new}^*$  is

$$K = [ -1.2732 \quad -0.8923 \quad -1.8967 \quad -1.8145 ]$$

Searching for three independent  $\epsilon$ 's can be a tedious job, particularly if the number of objectives is rather large. Here, to show the effect of independent  $\epsilon$ 's, we fix one of the  $\epsilon$ 's, starting with the value obtained above (i.e., when all were set equal to one another), and search for the other two  $\epsilon$ 's, which are assumed to be equal. The best result obtained is  $\gamma^* = 0.8104$  for  $\epsilon_2 = 0.0802$  and  $\epsilon_1 = \epsilon_3 = 0.1292$ . Searching for three independent  $\epsilon$  leads on a slightly better result (about 3%), which might be enhanced if a full – or more thorough – search is done, but the computation burden will be significant.

### 3.3 Output Feedback Case

In this section, we use the dilated MIs in designing a full order dynamic output feedback compensator for the same saturation problem. Therefore, the controller is

$$\begin{cases} \dot{x}_c = A_c x_c + B_c y \\ u = C_c x_c \end{cases} \quad (14)$$

where  $A_c$  is of the same order as the system matrix  $A$ . To reduce clutter of the equations, we dropped  $D_c$ . Applying this controller, the closed-loop system in (3) can easily be obtained with  $x = [x_p^T \quad x_c^T]^T$ .

In output feedback case, additional complications arise since a variety of transformations and manipulations are needed to set the problem into a convex search, often requiring auxiliary variables instead of the compensator matrices ( $A_c$ ,  $B_c$  and  $C_c$ ). The approach is reasonably well known and can be found in a variety of references (e.g., [4] and [14] among many). Here, most of the technical details are omitted but can be found in the references mentioned, though the outline is based on the approach used in [4] and [14]. Here, for the Lyapunov matrix in Algorithm 1, we use the structure

$$P = \begin{pmatrix} Y & -Y \\ -Y & S^{-1} + Y \end{pmatrix} > 0$$

which, as discussed in [14], can be done without any loss of generality. Therefore, Algorithm 1 can be expressed through the following lemma.

**Lemma 5 ([2])** *System (3) with disturbance  $w(t)$  satisfying  $w(t)^T w(t) \leq w_{max}^2$  and with a output feedback compensator (14) is internally stable, never saturates and has a disturbance attenuation level  $\gamma_{con}$  if there exist  $Y > 0$ ,  $X > 0$ , and general matrices  $L$ ,  $F$  and  $E$  and a constant  $\alpha_{con} > 0$  such that*

$$\begin{pmatrix} \Pi & A + L^T & B_1 & X^T C_1^T + F^T D_{12}^T \\ \star & \Lambda & Y B_1 + E D_{21} & C_1 \\ \star & \star & -\gamma_{con} I & D_{11} \\ \star & \star & \star & -\gamma_{con} I \end{pmatrix} < 0$$

$$\begin{pmatrix} \Pi + \alpha_1 X & A + L^T + \alpha_{con} I & B_1 \\ \star & \Lambda + \alpha_{con} Y & Y B_1 + E D_{21} \\ \star & \star & -\alpha_{con} I \end{pmatrix} < 0$$

$$\begin{pmatrix} -X & -I & -F^T \\ \star & -Y & 0 \\ \star & \star & -\frac{u_{lim}^2}{\omega_{max}^2} I \end{pmatrix} < 0$$

where  $\Pi = \mathbf{He}(AX + B_2 Y)$  and  $\Lambda = \mathbf{He}(A^T Y + E C_2)$ . Then for  $S = X - Y^{-1}$ , one representation of the controller matrices is:

$$C_c = F S^{-1}, \quad B_c = -Y^{-1} E$$

$$A_c = (A - B_c C_2) X S^{-1} + B_2 C_c - Y^{-1} L S^{-1}$$

For dilated MIs, also we need to do some manipulations to expand them into appropriate convex or near convex forms. We start with dilated MI for  $L_2$  gain (5). As in the state feedback multi-objective solution, eventually we are going to use common  $G$ , the key slack variable introduced by dilation. As a result, for simplification, the index  $i$  of  $G_i$  ( $i = 1, 2, 3$ ) is dropped. Let us call  $G^{-1} = H$ . Note that based on the structure of dilated MIs,  $G$  is invertible. By pre- and post-multiplying (5) by

$$\text{Diag} [ T^T H^T \quad I \quad I \quad T^T H^T ]$$

and its transpose, respectively, we obtain

$$\begin{pmatrix} \Phi_{11} & T^T H^T B_{cl} & T^T C_{cl}^T & \Phi_{14} \\ \star & -\gamma_2 I & D_{cl}^T & 0 \\ \star & \star & -\gamma_2 I & -2\epsilon_1 (C_{cl} T) \\ \star & \star & \star & -2\epsilon_1 \Phi_{44} \end{pmatrix} < 0 \quad (15)$$

where

$$\begin{aligned} \Phi_{11} &= T^T H^T X_1 H T + \mathbf{He}(T^T H^T A_{cl} T) - \frac{1}{2} \Phi_{44} \\ \Phi_{14} &= -T^T (H^T X_1 H + H - 2\epsilon_1 (H^T A_{cl} - \frac{1}{2} H^T)) T \\ \Phi_{44} &= \mathbf{He}(T^T H^T T) \end{aligned}$$

Here,  $T$  is an auxiliary matrix used for the additional transformations that are needed in the output feedback synthesis problem.

We partition  $H$  and  $G$  into the following forms with each submatrix having the dimension  $n \times n$

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \quad G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}. \quad (16)$$



As mentioned before,  $G$  is invertible, so the sub-matrices of this matrix are also invertible (by invoking a small perturbation if necessary [15]). Now, to turn dilated MIs into near convex form, let's consider the the auxiliary transformation matrix  $T$  as

$$T = \begin{pmatrix} G_{11} & I \\ G_{21} & 0 \end{pmatrix}.$$

To simplify notations, we call  $G_{11} \equiv R$  and  $H_{11} \equiv Y$ . Finally, let us call

$$T^T H X_1 H T = M_1 = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{13} \end{pmatrix}; (M_1 = M_1^T)$$

and

$$F = C_c G_{21}, \quad E = H_{21}^T B_c, \quad V = R^T Y + G_{21}^T H_{21} \\ L = (Y^T A + E C_2) R + Y^T B_2 F + H_{21}^T A_c G_{21}$$

These are the new variables introduced to turn this MIs into convex form (similar to  $G$ ,  $F$ ,  $L$  used in the standard approach. Also to save space, for  $i = 1, 2$ , we use

$$\Pi = AR + B_2 F - \frac{1}{2}R, \quad \Lambda = Y A + E C_2 - \frac{1}{2}Y^T \\ \Omega = -\frac{1}{2}(I + V) + A + L^T, \quad \Delta_i = -M_{i1} + R^T - 2\epsilon_i \Pi \\ \Sigma_i = -M_{i3} + Y - 2\epsilon_i \Lambda, \quad \Gamma_i = -M_{i2} + V - 2\epsilon_i(A - \frac{1}{2}I) \\ \Upsilon_i = -M_{i2}^T + I - 2\epsilon_i(L - \frac{1}{2}V^T)$$

Naturally, the same manipulation can be conducted on MI (10) and (12), albeit with a bit less clutter. Considering above matrices and definitions, the entries of the three MIs can be expanded in detail as in the following lemma which expresses Algorithm 2 for dynamic output feedback compensator.

**Lemma 6** *System (3) with disturbance  $w(t)$  satisfying  $w^T(t)w(t) \leq w_{max}^2$  and with a output feedback compensator (14) is internally stable, never saturates and has a disturbance attenuation level  $\gamma_{new}$  if there exist square matrices  $R, S, V$  and symmetric matrices  $Y, M_1, M_2$*

$$M_1 = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{13} \end{pmatrix}, \quad M_2 = \begin{pmatrix} M_{21} & M_{22} \\ M_{22}^T & M_{23} \end{pmatrix}$$

and general matrices  $L, E, F$ , and constant  $\alpha_{new} > 0$ , and small positive value  $\epsilon_i < 1$  ( $i = 1, 2, 3$ ) such that

$$\begin{pmatrix} M_{11} + \mathbf{He}(\Pi) & M_{12} + \Omega & B_1 \\ \star & M_{13} + \mathbf{He}(\Lambda) & Y^T B_1 + E D_{21} \\ \star & \star & -\gamma_{new} I \\ \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \\ R^T C_1^T + F^T D_{12}^T & \Delta_1 & \Gamma_1 \\ C_1^T & \Upsilon_1 & \Sigma_1 \\ D_{11}^T & 0 & 0 \\ -\gamma_{new} I & -2\epsilon_1(C_1 R + D_{12} F) & -2\epsilon_1 C_1 \\ \star & -2\epsilon_1 \mathbf{He}(R) & -2\epsilon_1(I + V) \\ \star & \star & -2\epsilon_1 \mathbf{He}(Y) \end{pmatrix} < 0$$

$$\begin{pmatrix}
M_{21} + \mathbf{He}(\frac{\alpha_{new}}{2}R + \Pi) & M_{22} + \frac{\alpha_{new}}{2}(I + V) + \Omega & \\
\star & M_{23} + \mathbf{He}(\frac{\alpha_{new}}{2}Y + \Lambda) & \\
\star & \star & \\
\star & \star & \\
\star & \star & 
\end{pmatrix}
\begin{pmatrix}
B_1 & \Delta_2 - 2\epsilon_2(\frac{\alpha_{new}}{2}R) & \Gamma_2 - 2\epsilon_2(\frac{\alpha_{new}}{2}I) \\
Y^T B_1 + ED_{21} & \Upsilon_2 - 2\epsilon_2(\frac{\alpha_{new}}{2}V^T) & \Sigma_2 - 2\epsilon_2(\frac{\alpha_{new}}{2}Y^T) \\
-\alpha_{new}I & 0 & 0 \\
\star & \mathbf{He}(-2\epsilon_2 R) & -2\epsilon_2(I + V) \\
\star & \star & \mathbf{He}(-2\epsilon_2 Y)
\end{pmatrix} < 0$$

$$\begin{pmatrix}
M_{21} - \mathbf{He}(R) & M_{22} - I - V & -F^T \\
\star & M_{23} - \mathbf{He}(Y) & 0 \\
\star & \star & -\frac{u_{max}^2}{\omega_{max}^2}I \\
\star & \star & \star \\
\star & \star & \star
\end{pmatrix}
\begin{pmatrix}
-M_{21} + R^T + 2\epsilon_3 R & -M_{22} + V + 2\epsilon_3 I \\
-M_{22}^T + I + 2\epsilon_3 V^T & -M_{23} + Y + 2\epsilon_3 Y^T \\
F & 0 \\
\mathbf{He}(-2\epsilon_3 R) & -2\epsilon_3(V + I) \\
\star & \mathbf{He}(-2\epsilon_3 Y)
\end{pmatrix} < 0$$

Then for invertible  $G_{21}$  and  $H_{21}$  deduced by matrix factorization of  $G_{21}^T H_{21} = V - R^T Y$  as mentioned in [16], the controller matrices are as follows:

$$C_c = FG_{21}^{-1}, \quad B_c = H_{21}^{-T}E$$

$$A_c = H_{21}^{-T}(L - Y^T AR + EC_2 R + YB_2 F)G_{21}^{-1}$$

As mentioned above, invertible  $G_{21}$  and  $H_{21}$  can be deduced by matrix factorization of  $G_{21}^T H_{21} = V - R^T Y$ . As indicated in [16], this deduction is always possible and if necessary we can use perturbation. In our numerical examples, when possible, we obtain  $G_{21}$  and  $H_{21}$  by picking  $H_{21} = Y$  and therefore having  $G_{21} = Y^{-T}V^T - R$ .

As before, to avoid excessive computational cost in the search for  $\epsilon_1$ ,  $\epsilon_2$ , and  $\epsilon_3$  in Lemma (6), we can use same  $\epsilon$  for all MIs.

### 3.4 Numerical example: Output Feedback

Consider the same numerical example as for the state feedback, this time with  $k = 0.4$ ,  $f = 0.04$ ,  $u_{lim} = 100$  and with  $y^T = [x_1 \ x_2]$ .

This system is expected to withstand disturbances with peak bound of  $w_{max} = 5$ . Using the conventional approach, the resulting controller has  $A_c$  and  $B_c$  matrices of order  $10^5$  and

$$C_c|_{con} = [ -12.5082 \quad -18.5711 \quad -5.8239 \quad -42.4745 ]$$

The induced  $L_2$  gain of this system under this controller is  $\gamma_{con}^* = 1.7636$ .

This problem is also solved with the new dilated matrix inequalities. We start with the same  $\epsilon$  in all three dilated matrix inequalities, and we use Golden-ratio method in search over  $\epsilon$ . This, after 10 iteration, leads to  $\gamma_{new}^* = 1.5029$  at  $\epsilon = 0.0231$ , which is about 15% improvement, over the conventional method. The corresponding controller has  $A_c$  and  $B_c$  matrices of order  $10^4$  and

$$C_c|_{new} = [ -11.5857 \quad -17.0741 \quad -5.5596 \quad -38.7530 ]$$

Next, we try independently varying  $\epsilon$ 's. To limit the cost of searching for three independent  $\epsilon$ 's, we have fixed one of the  $\epsilon$ 's in a certain value and search for the other two  $\epsilon$ 's which are assumed to be equal. The best result obtained is  $\gamma^* = 1.2746$  for  $\epsilon_2 = 0.0181$  and  $\epsilon_1 = \epsilon_3 = 0.1618$ . Thus, letting  $\epsilon$ 's vary independently has a significant effect at the results, an improvement of about 27% over the conventional method of solving the problem ( $\gamma_{con}^* = 1.7636$ ).

## 4 CONCLUSIONS

We presented new dilated matrix inequalities for Bounded Real MI, invariant set MI and constraint MI. The structure of these dilated MIs, in which system matrices are separated from Lyapunov matrices, allows us to use different Lyapunov matrices for different objective in multi-objective problems or different parameter values in robust synthesis problems. The new approach is guaranteed to achieve results which are better or equal to the ones obtained from the standard multi-objective setting. The synthesis results, for both state feedback and output feedback problems, are demonstrated through an example.

## References

- [1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM studies in Applied Mathematics, 1998.
- [2] I.E. Köse and F. Jabbari, Scheduled Controller for Linear Systems with Bounded Actuators, *Automatica*, vol. 39, 2003, pp 1377-1387.
- [3] M. Chilali and P. Gahinet,  $H_\infty$  Design with Poleplacement Constraints: an LMI Approach, *IEEE Trans. Automat. Contr.*, vol. 41, No.3, 1996, pp 358-363.
- [4] C. Scherer, P. Gahinet and M. Chilali, Multiobjective Output-Feedback Control via LMI Optimization *IEEE Trans. Automat. Contr.*, vol. 42, No.7, 1997, pp 896-911.
- [5] P. Gahinet and P. Apkarian, A Linear Matrix Inequalities Approach to  $H_\infty$  Control, *Int. J. Nonlinear Robust Contr.*, vol. 4, 1994, pp. 421-448.
- [6] M.C. Oliveira, J. Bernussou, and J.C. Geromel, A New Discrete-Time Robust Stability Condition, *Systems & Contr. Lett.*, vol. 37, 1999, pp 261-265.
- [7] M.C. Oliveira, J.C. Geromel and J. Bernussou, "An LMI Optimization Approach to Multi-objective Controller Design for Discrete-Time Systems", *Proc. Conf. Decision and Contr.*, 1999, pp 3611-3616.
- [8] Y. Ebihara and T. Hagiwara, "New Dilated LMI Characterizations For Continuous-Time Control Design and Robust Multi-Objective Control", *Proc. American Contr. Conf.*, Anchorage, 2002, pp 47-52.
- [9] Y. Ebihara, T. Hagiwara, New Dilated LMI Characterizations for Continuous-Time Control Multi-Objective controller Synthesis, *Automatica*, vol. 40, 2004, pp 2003-2009.
- [10] Y. Ebihara and T. Hagiwara, A Dilated LMI Approach to Robust Performance Analysis of Linear Time-Invariant Uncertain Systems, *Automatica*, vol. 41, 2005, pp 1933-1941.
- [11] Y. Hamada, I. Yamaguchi and T. Kida, "Gain Scheduling Controller Synthesis with Spline-Type Parameter-Dependent Quadratic Forms via Dilated Linear Matrix Inequities", *Proc. 44th IEEE Conf. on Decision and Control, and the European Control Conf.*, Spain, Dec. 2005., pp. 1313-1318.

- [12] J. Wei and L. Lee, "Further Improvement on LMI Representations for the Analysis and Design of Continuous-Time System with Polytropic Type Uncertainty", *Proc. of 5th Asian Contr. Conf.*, 2004 pp p.1-p.7.
- [13] U. Shaked, Improved LMI Representation for the Analysis and Design of Continuous-Time Systems with Polytropic Uncertainty", *IEEE Trans. AC*, vol. 46, No.4, 2001, pp 652-656.
- [14] F. Jabbari, Output Feedback Controllers for Systems with Structured Uncertainty, *IEEE Trans. AC*, vol. 42, No. 5, 1997, pp 715-719.
- [15] H.D. Tuan, P. Apkarian and T.Q. Nguyen, Robust and Reduced-Ordered Filtering: New LMI-Based Characterizations and Methods, *IEEE Trans. Signal Processing*, vol. 49, No.12, 2001, pp 2975-2984.
- [16] P. Apkarian, H.D. Tuan and J. Bernussou, Continuous-Time Analysis, Eigenstructure Assignment and  $H_2$  Synthesis with Enhanced Linear Matrix Inequalities (LMI) Characterizations", *IEEE Trans. AC*, vol. 46, No.12, 2001, pp 1941-1946.